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Operator Theoretic Invariants and the Enumeration Theory of Pólya and de Bruijn

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ABSTRACT

It has been shown by M. Marcus and others that, in regard to combinatorial matrix functions and combinatorial inequalities, it is frequently fruitful to pass immediately from the consideration of permutations to the consideration of their tensor representations. Such an approach embeds the combinatorial arguments into the framework of linear algebra and frequently results in deeper theorems. It is interesting to note that certain basic combinatorial identities concerned with pattern enumeration and combinatorial generating functions can also be put into this framework. In this paper we consider one possible way of doing this.

1. INTRODUCTION

One of the most frequently quoted theorems in enumerative combinatorial analysis is the so called Fundamental Theorem of Pólya [1, 2, 3, 5, 8]. More recently, N. G. de Bruijn has given several interesting and useful extensions of this theorem [1–3]. In this paper we show the connection between this theory and the operator theoretic approach to the study of symmetries of H. Weyl [4, 9].

In particular we show that the identity which constitutes Pólya's Theorem arises as the expression of equality between the traces of two representations of the same symmetry operator with respect to two different bases for the underlying tensor space. The generalizations of Pólya's Theorem due to de Bruijn [2, 3] arise in the same manner with respect to a slightly larger class of operators.

2. BASIC DEFINITIONS AND STATEMENTS OF THEOREMS

Let R and D be finite sets and let R^D denote the set of all functions with domain D and range R. Assume that $|R| = m$ and $|D| = n$

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where, for any finite set S, $|S|$ denotes the number of elements in S. For terminology related to permutation groups see [10]. Let G and H be permutation groups acting on D and R, respectively, and consider the power group $(H; G)$. This group (of order $|H||G|$) acts on R^D in the sense that, if $\sigma \in G$, $\tau \in H$, and $(\tau; \sigma) \in (H; G)$, then, for any $\gamma \in R^D$, $(\tau; \sigma)\gamma = \tau\gamma\sigma^{-1}$ where $\tau\gamma\sigma^{-1}(d) = \tau(\gamma(\sigma^{-1}(d)))$ for all $d \in D$ (i.e., $\tau\gamma\sigma^{-1}$ denotes ordinary function composition). Assume without loss of generality that $R = Z_m = \{1,..., m\}$ and $D = Z_n$. Let V denote an m-dimensional vector space over a field F of characteristic zero. Let $U = \otimes^n V$ denote the n-th tensor product of V. Then for any $\gamma \in R^D$ the set of tensors $\beta = \{e_\gamma : \gamma \in R^D\}$ forms a basis for the space U where

$$e_\gamma = e_{\gamma(1)} \otimes \cdots \otimes e_{\gamma(n)} \quad [7].$$

For each $(\tau; \sigma) \in (H; G)$ define an operator $P(\tau; \sigma) : U \to U$ by $P(\tau; \sigma)\, e_\gamma = e_{\tau\gamma\sigma^{-1}}$. One readily checks that $P(\tau; \sigma)$ is non-singular and in fact a permutation operator with respect to the basis β (i.e., simply permutes the basis elements among themselves). Observe also that $P(\tau_1 ; \sigma_1)\, P(\tau_2 ; \sigma_2) = P(\tau_1\tau_2 ; \sigma_1\sigma_2)$ and hence $P(\tau; \sigma)$ is a representation of $(H; G)$. Define a *symmetry operator* $T_{(H;G)}$ on U as follows:

$$T_{(H;G)} = \frac{1}{|H||G|} \sum_{(H;G)} \alpha(\tau; \sigma)\, P(\tau; \sigma), \tag{1}$$

where α is any homomorphism of $(H; G)$ into the multiplicative group of the field F. Using the fact that both α and P are representations of $(H; G)$ one easily verifies that $T_{(H;G)}$ is idempotent. In this paper we discuss only the case $\alpha(\tau; \sigma) \equiv 1$. The more general care relates to an enumeration theory (which will be discussed in another paper) that excludes structures with certain symmetries related to the character α. Thus we assume that

$$T_{(H;G)} = \frac{1}{|H||G|} \sum_{(H;G)} P(\tau; \sigma). \tag{2}$$

We call $T_{(H;G)}$ the symmetry operator associated with $(H; G)$.

We now introduce the notion of a *weighted symmetry operator*. Let Δ be a system of distinct representatives for the orbits of $(H; G)$ in R^D. Let $W : R^D \to F$ be a function from R^D to the field F which is constant on these orbits. Thus W assumes at most $|\Delta|$ values.

LEMMA 1. *Let U_α be the space spanned by all tensors $\{e_\gamma : W(\gamma) = \alpha\}$. Then $U = \oplus_{\alpha \in F} U_\alpha$ and for each α, U_α is an invariant subspace for the family of operators $\{P(\tau; \sigma) : (\tau; \sigma) \in (H; G)\}$ and hence for the symmetry operator $T_{(H;G)}$.*

DEFINITION 1. Let $T^{\alpha}_{(H;G)}$ and $P_{\alpha}(\tau; \sigma)$ denote the restriction of $T_{(H;G)}$ and $P(\tau; \sigma)$, respectively, to the subspace U_{α}. A *weighted permutation operator* $P_W(\tau; \sigma)$ is an operator of the form

$$P_W(\tau; \sigma) = \bigoplus_{\alpha \in F} \alpha P_{\alpha}(\tau; \sigma) \tag{3}$$

and a *weighted symmetry operator* is an operator of the form

$$T^W_{(H,G)} = \bigoplus_{\alpha \in F} \alpha T^{\alpha}_{(H;G)}. \tag{4}$$

LEMMA 2. *The trace of $P_W(\tau; \sigma)$ is given by $\sum_{\gamma}^{(\tau;\sigma)} W(\gamma)$ where $\sum_{\gamma}^{(\tau;\sigma)} W(\gamma)$ denotes the sum over all γ fixed by $(\tau; \sigma)$.*

THEOREM 1. *We have that*

$$T^W_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} P_W(\tau; \sigma). \tag{5}$$

Furthermore, the set of vectors $\{T_{(H;G)}(e_{\gamma}) : \gamma \in \Delta\}$ where Δ is a system of distinct representatives for the orbits of $(H; G)$ in R^D form a basis of eigenvectors for the symmetry class, range $T^W_{(H;G)}$, with corresponding eigenvalues $\{W(\gamma) : \gamma \in \Delta\}$.

COROLLARY 1. *The identity*

$$\operatorname{tr} T^W_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} \operatorname{tr} P_W(\tau; \sigma) \tag{6}$$

becomes

$$\sum_{\gamma \in \Delta} W(\gamma) = \frac{1}{|H|\,|G|} \sum_{(H;G)} \sum_{\gamma}^{(\tau;\sigma)} W(\gamma). \tag{7}$$

Corollary 1 is an immediate consequence of Lemma 2 and Theorem 1. The identity (7) is due to de Bruijn [1, 2] and is a generalization of Pólya's Theorem [1, 8].

To obtain Pólya's Theorem from (7) we must define the *cycle index polynomial* of a permutation group G. For any permutation σ acting on D let k_i^{σ} denote the number of cycles of length i in the disjoint cycle decomposition of σ (the n-tuple $(k_1^{\sigma},..., k_n^{\sigma})$ is called the "type" of σ). Let $P_G(x_1,..., x_n)$ denote a polynomial in n variables (over F) defined by

$$P_G(x_1,..., x_n) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{k_1^{\sigma}} \cdots x_n^{k_n^{\sigma}}. \tag{8}$$

In the case of Pólya's Theorem we consider only groups of the form $(I; G)$ where $I = \{e\}$ is the identity group. We identify $(I; G)$ with G in the natural manner. Let $\omega : R \to F$ and define

$$W(\gamma) = \prod_{d \in D} \omega(\gamma(d)). \tag{9}$$

It is easy to see that W is constant on the orbits of $(I; G) \equiv G$ in R^D. Also, it is clear that γ is fixed by $\sigma \in G$ if and only if γ is constant on the cycles of σ. For such a γ, if d is an element in a cycle of length i of σ, then the expression $\omega(\gamma(d))^i$ occurs as a factor in $W(\gamma)$ and, given any $r \in R$, $\xi(d) = r$ for some $\xi \in R^D$ fixed by σ. In this manner, considering all such γ fixed by σ, one concludes that

$$\sum_{\gamma}^{\sigma} W(\gamma) = \left(\sum_{r \in R} \omega(r)\right)^{k_1^\sigma} \left(\sum_{r \in R} \omega(r)^2\right)^{k_2^\sigma} \cdots \left(\sum_{r \in R} \omega(r)^n\right)^{k_n^\sigma}, \tag{10}$$

where "$\sum_{\gamma}^{\sigma}$" denotes the sum over all γ fixed by $\sigma \equiv (e; \sigma)$ (i.e., $\gamma\sigma^{-1} = \gamma$).

Thus (7), (8), and (10) imply that

$$\sum_{\gamma \in \Delta} W(\gamma) = P_G \left(\sum_{r \in R} \omega(r), \sum_{r \in R} [\omega(r)]^2, \dots, \sum_{r \in R} [\omega(r)]^n\right). \tag{11}$$

The identity (11) constitutes Pólya's Theorem [1, 8]. In the combinatorial literature W (and ω) are called "weight functions" and $\sum_{\gamma \in \Delta} W(\gamma)$ is called the "pattern inventory."

In terms of weighted symmetry operators $T^W_{(I;G)} \equiv T_G{}^W$, (11) becomes

$$\operatorname{tr} T_G{}^W = P_G \left(\sum_{r \in R} \omega(r), \sum_{r \in R} [\omega(r)]^2, \dots, \sum_{r \in R} [\omega(r)]^n\right). \tag{12}$$

The identities (11) and (12) can be extended easily to the case of a finite collection $\{G_s : s \in \Delta'\}$ of groups acting on D. Let $W : R^D \to F$ be as in (9). Consider $T^W_{(I;G_s)} \equiv T^W_{G_s}$ and the direct sum $\bigoplus_{s \in \Delta'} T^W_{G_s}$. From (12) we see that

$$\operatorname{tr}\left[\bigoplus_{s \in \Delta'} T^W_{G_s}\right] = \sum_{s \in \Delta'} P_{G_s} \left(\sum_{r \in R} \omega(r), \dots, \sum_{r \in R} [\omega(r)]^n\right) \tag{13}$$

or

$$\sum_{s \in \Delta'} \sum_{\gamma \in \Delta_s} W(\gamma) = U \left(\sum_{r \in R} \omega(r), \dots, \sum_{r \in R} [\omega(r)]^n\right), \tag{14}$$

where Δ_s is a system of distinct representatives for the orbits of G_s in R^D and

$$U(x_1, ..., x_n) = \sum_{s \in \Delta'} P_{G_s}(x_1, ..., x_n). \tag{15}$$

In [3] de Bruijn has given several interesting combinatorial applications of (14). For example, let D be a finite set of points and let S be the set of all graphs with the points D as vertices. Let R be a finite set of colors or labels. We may regard the set $S^* = S \times R^D$ as the set of labeled or colored graphs. Let Ψ be the standard representation of G as a group of permutations $\bar{G}$ on S (if s is a graph then $\Psi(\sigma)\, s$ is the graph such that d_1 and d_2 are joined if and only if $\sigma^{-1}(d_1)$ and $\sigma^{-1}(d_2)$ are joined in s). For each s let $G_s = \Psi^{-1}(\bar{G}_s)$ where $\bar{G}_s$ is the stabilizer subgroup of $\bar{G}$ for the element s. Let Δ' be a system of distinct representatives for the orbits of $\bar{G}$ in S. The identity (14) may be used in connection with enumeration problems involving the set S^* of colored or labeled graph structures [3].

Finally, we make some remarks concerning the range of the weight function W. We have assumed that $W: R^D \to F$ where F is a field of characteristic zero. As pointed out by the Bruijn [1, 3] identities (7), (11), and (14) remain valid if $W: R^D \to A$ where A is any commutative ring (which somehow allows multiplication by rationals). However in practically all (perhaps all?) applications of interest it seems that A is such that it can be extended to a field F of characteristic zero (integers to the field of rationals, variables to the field of rational functions, etc.) or A is the homomorphic image of such a ring (integers modulo k for example). In either case it suffices to consider a field F of characteristic zero in deriving identities (7), (11), and (14). With certain conceptual and technical disadvantages the operator theoretic approach of this paper will extend to the case in which V is a commutative A-module, however.

3. Proofs

For basic definitions concerning permutation groups and multilinear algebra see [10] and [7], respectively. For a thorough discussion of symmetry operators $T_{(H:G)}$ in the case $H = I$ (the identity group) and F is the field of complex numbers see Marcus and Minc [6].

Proof of Lemma 1: Clearly if $\alpha_1 = W(\gamma_1) \neq W(\gamma_2) = \alpha_2$ then $\gamma_1 \neq \gamma_2$. Hence the spanning sets for U_{α_1} and U_{α_2} are disjoint. Since the set $\beta = \{e_\gamma : \gamma \in R^D\}$ forms a basis for $U = \otimes^n V$ we have immediately that $U = \bigoplus_{\alpha \in R} U_\alpha$. Since W is constant on the orbits of the power group $(H; G)$ acting on R^D we have that $W(\tau\gamma\sigma^{-1}) = W(\gamma)$ for all $(\tau; \sigma) \in (H; G)$ implies that if $e_\gamma \in U_\alpha$ then $P(\tau; \sigma)\, e_\gamma \in U_\alpha$. Thus U_α is invariant relative to the family of operators $\{P(\tau; \sigma) : (\tau; \sigma) \in (H; G)\}$. Since

$$T_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} P(\tau; \sigma),$$

U_α is also an invariant subspace of $T_{(H;G)}$. This completes the proof of Lemma 1.

Proof of Lemma 2: To prove Lemma 2 let $\beta = \{e_\gamma : \gamma \in R^D\}$ denote the canonical basis for the tensor space U ordered lexicographically. Consider the matrix $M = [P_W(\tau; \sigma)]_\beta^\beta$ of the weighted symmetry operator $P_W(\tau; \sigma)$ with respect to the basis β. Let $e_\gamma \in \beta$. Notice that M has a non-zero entry on the diagonal corresponding to e_γ if and only if $P_W(\tau; \sigma)\, e_\gamma$ has a non-zero component along e_γ. But $P_W(\tau; \sigma)\, e_\gamma = W(\gamma)\, P(\tau; \sigma)\, e_\gamma$. Since $P(\tau; \sigma)$ is a permutation operator relative to the basis β (i.e., simply permutes the vectors of the basis β among themselves) we can conclude that $P_W(\tau; \sigma)\, e_\gamma$ has a non-zero component along e_γ if and only if $W(\gamma) \neq 0$ and $P(\tau; \sigma)\, e_\gamma = e_\gamma$ or $\tau\gamma\sigma^{-1} = \gamma$. In this case the diagonal entry is simply $W(\gamma)$. Thus the only non-zero diagonal entries of M correspond to those γ which are fixed by $(\tau; \sigma)$ and these have the values $W(\gamma)$. Thus

$$\operatorname{tr} M = \sum_{\gamma}^{(\tau;\sigma)} W(\gamma)$$

as asserted by Lemma 2.

Proof of Theorem: Let $T^\alpha_{(H;G)}$ and $P_\alpha(\tau; \sigma)$ denote the restrictions of $T_{(H;G)}$ and $P(\tau; \sigma)$ to U_α. Since U_α is invariant relative to $T_{(H;G)}$ and $P(\tau; \sigma)$ we have from (2) that

$$T^\alpha_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} P_\alpha(\tau; \sigma),$$

and hence

$$\bigoplus_{\alpha \in F} \alpha T^\alpha_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} \bigoplus_{\alpha \in F} \alpha P_\alpha(\tau; \sigma)$$

or

$$T^W_{(H;G)} = \frac{1}{|H|\,|G|} \sum_{(H;G)} P_W(\tau; \sigma),$$

which proves (5).

Consider now the set $\beta^* = \{T_{(H;G)}(e_\gamma) : \gamma \in \Delta\}$. Since

$$\{T_{(H;G)}(e_\gamma) : \gamma \in R^D\}$$

spans the range of $T_{(H;G)}$ so does β^* (observe that if γ_1 and γ_2 are in the same orbit of $(H; G)$ in R^D then $T_{(H;G)}(e_{\gamma_1}) = T_{(H;G)}(e_{\gamma_2})$). Also β^* is a linearly independent set. Consider

$$\sum_{\gamma \in \Delta} d_\gamma T_{(H;G)}(e_\gamma) = \theta \text{ (zero vector).} \tag{16}$$

Let $\{f_i\}$ $i = 1,..., n$ be a basis of V^* dual to $\{e_i\}$ $i = 1,..., n$. Let $f_\gamma = f_{\gamma(1)} \otimes \cdots \otimes f_{\gamma(n)}$ be in $\otimes^n V^*$. Recall that

$$f_\gamma(e_\beta) = \prod_{t=1}^{n} f_{\gamma(t)}(e_{\beta(t)}).$$

Now let K_α denote the number of elements in the stabilizer subgroup of $(H; G)$ corresponding to α. Then

$$f_\alpha(T_{(H;G)}(e_\gamma)) = (| H | | G |)^{-1} K_\alpha$$

if α and γ are in the same orbit and zero otherwise. Evaluating f_α, $\alpha \in \Delta$ at the zero vector (16) gives

$$f_\alpha\left(\sum_{\gamma \in \Delta} d_\gamma T_{(H;G)}(e_\gamma)\right) = (| H | | G |)^{-1} K_\alpha d_\alpha = 0. \tag{17}$$

Thus $d_\alpha = 0$ for $\alpha \in \Delta$ and β^* is a basis for range $T_{(H;G)}$.

To check that $T_{(H;G)}(e_\gamma)$, $\gamma \in \Delta$, is an eigenvector of $T^W_{(H;G)}$ note that $T_{(H;G)}(e_\gamma)$ is in U_α if $W(\gamma) = \alpha$.

Thus

$$\begin{aligned} T^W_{(H;G)}(T_{(H;G)}(e_\gamma)) &= W(\gamma)\, T^2_{(H;G)}(e_\gamma) \\ &= W(\gamma)\, T_{(H;G)}(e_\gamma) \end{aligned}$$

since $T_{(H,G)}$ is idempotent (as may be seen directly from (2) and the fact that $P(\tau; \sigma)$ is a representation of $(H; G)$). Thus $T_{(H;G)}(e_\gamma)$ is an eigenvector of $T^W_{(H;G)}$ with corresponding eigenvalue $W(\gamma)$. This completes the proof of Theorem 1.

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